A Variance Equality Test of the ICAPM on Philippine Stocks:

Post Asian Financial Crisis Period

Rodolfo Q. Aquino*

Abstract

The study examines whether Fama's discrete version of Merton's intertemporal CAPM (ICAPM) can explain the cross sectional variability of Philippine stock returns after the onset of the Asian financial crisis in July 1997.Unanticipated changes in foreign exchange rate is used as a state variable of hedging concern to investors in addition to changes in market risk premium. The relationship between Fama's multifactor minimum-variance (MMV) portfolio to the Markowitz minimum- variance (MV) portfolio is characterized in terms of the equality of the return variances for the same expected return. A test due to Basak, Jagannathan and Sun is then used to test the equality of the return variance of a derived tangency portfolio along the MMV frontier to an MV portfolio with the same sample mean return. The results do not reject the ICAPM during the period covered by the study.

Correspondence:

Rodolfo Q. Aquino
College of Business Administration
University of the Philippines
Diliman, Quezon City
PHILIPPINES

E-mail: rodolfo.aquino@up.edu.ph

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I. Introduction

Tests of the capital asset pricing model (CAPM) on Philippine stocks for periods prior to the onset of the Asian financial crisis in July 1997 have generally been supportive of the theory (for example, Yu, 2002; Aquino, 2002-2003; and De Ocampo, 2004). However, tests covering periods after the onset of the crisis do not support the CAPM predictions except only the existence of a positive relationship between risk and return. In particular, the expected market risk premium appears to be negative in these studies and along the negative slope of the Markowitz minimum-variance (MV) frontier contrary to the assumption of mean-variance optimizing investors within the context of the model. Also, other factors appear to influence cross-sectional variation in expected excess returns.

This study explores the ability of Merton's intertemporal CAPM (ICAPM) to explain the empirical data on Philippine stock returns (July 1997 – December 2001). In particular, Fama (1996) noted that, under the ICAPM, the expected market risk premium can be negative. Fama's discrete version of the ICAPM is especially useful in this regard because it can naturally depict the relationships of portfolios in mean-variance (more exactly, mean-standard deviation) space.

In the ICAPM, state variables, in addition to market movements, are postulated to affect cross-sectional difference in asset returns in a multi-period setting. The effects of state variables on returns result from the desire of investors to hedge against changes in investment opportunities. Merton (1973) suggests that these changes are captured by changes in the real interest rate. In this study, a measure of the subsequent volatile changes in the peso to U.S. dollar exchange rate is used as the instrumental variable that predicts changes in investment opportunities through changes in the real exchange rate and interest parity condition. Prior to the Asian financial crisis, there has been a long period of relative exchange rate stability (Gochoco-Bautista and Bautista, 2004). However, the crisis changed all that as the Philippine peso, as most Asian currencies, were removed from what is generally considered to be a *de facto* peg.

This study is organized as follows: Section II discusses Fama's discrete version of Merton's ICAPM and some characterizations of the multifactor minimum variance

(MMV) frontier. It is shown that if the ICAPM holds, a MMV portfolio exists that is also MV in the sense of Markowitz. This coincidence is characterized by the equality of portfolio return variances for the same expected return. Thus, a natural test of the ICAPM that can be used is the test of equality of portfolio return variances due to Basak, Jagannathan and Sun (2002, hereinafter BJS). Section III discusses the data and testing methodology used in the study. Section IV presents the test results of the ICAPM on Philippine stock returns during the period covered by the study. Section V concludes the study.

II. Fama's Discrete ICAPM

In this section, the mathematics of Fama's (1996) MMV frontier will be reviewed and some characterizations of the frontier in mean-variance space established.

Like Merton (1973), Fama assumes multivariate normality of returns and state variables. This implies that the relationship between returns and state variables can be described as:

$$R = E + Bk + \varepsilon \qquad (1)$$

where R is the N x 1 vector of returns on the N assets, E = E(R), $B = [b_1 | b_2 | \ldots | b_S]$ is the N x S matrix of factor loadings on the state variables, k is the S x 1 vector of state variables of hedging concern to the investors such that E(k) = 0, and ε is the N x 1 vector of random errors such that E(ε) = 0 and cov (ε, k) = 0. Assume further that N > (S + 2) and B is of full rank (otherwise, one or more of the state variables can be dropped).

Let x be the N x 1 vector of proportions of the total investible resources invested on the N assets, i is an N x 1 vector of ones, V is the N x N nonsingular covariance matrix of returns, b_e is the S x 1 vector of target loadings on the state variables, r_p is the required expected return on the portfolio defined by x, and σ_{ip}^2 is the variance of the portfolio return. Analogous to the Markowitz MV frontier, the frontier of MMV portfolios can be obtained by solving the following optimization problem for various combinations of b_e and r_p (Fama, 1996):

$$\text{Minimize} \qquad \tfrac{1}{2}\sigma_{ip}^2 = \tfrac{1}{2} x^T V x \qquad (2)$$

Subject to: $B^T x = b_e$ (2.a)

$E^T x = r_p$ (2.b)

$i^T x = 1.$ (2.c)

The Markowitz MV frontier is obtained by solving (2) without constraint (2.a). For the solution to problem 2 to be consistent with the behavior of mean-variance optimizing investors, it must lie on the positive slope of the MMV frontier. Denote the portfolio return variances for the same r_p at the MV and MMV frontiers as σ^2_{mp} and σ^2_{ip}, respectively. Then, we have the following characterizations.

Proposition 1. $\sigma^2_{ip} \geq \sigma^2_{mp}$.

This proposition is implied by problem (2) but the formal proof is shown in the Appendix.

Proposition 2. If B is of full rank, then there exists at least one MMV portfolio that is also on the MV frontier.

The proof of this proposition is also in the Appendix.

If the N x 1 vector i is in the column space of B, the entire MV frontier is MMV and the problem reduces to that of the traditional one-period CAPM. In particular, any MV portfolio also meets the investor's target loadings for the state variables. This particular result is due to Huberman, Kandel and Stambaugh (1987, hereinafter HKS) and is shown as a corollary to Proposition 2 in the Appendix.

Mimicking portfolios are portfolios formed from the set of N risky assets that can be used in place of the factors in a multifactor model such as the arbitrage pricing theory (APT) model due to Ross (1976) or the ICAPM. HKS discuss three ways of forming mimicking portfolios. The one used here involves forming portfolios with returns that are maximally correlated with movements in the state variables, i.e.,

$$\text{maximize corr } (r_p, k_s) = \frac{\text{cov}\left(x^T R, k_s\right)}{\sigma\left(x^T R\right)\sigma\left(k_s\right)} = \frac{x^T b_s \sigma\left(k_s\right)}{\sigma\left(x^T R\right)} \quad (3)$$

where the last equality follows from (1). For different b_{es}, the sth element of b_e, consider the following optimization problem:

$$\begin{aligned} &\text{minimize} && \tfrac{1}{2} x^T V x \\ &\text{subject to} && b_s^T x = b_{es} \\ & && i^T x = 1. \end{aligned} \qquad (4)$$

Analogous to the single-factor CAPM, the solution to (3) is the tangency point of a line drawn from the origin to the efficient set of optimization problem (4). The solution is $x_s = V^{-1}b_s/i^TV^{-1}b_s$. The elements of the factor loading b_s can be estimated by ordinary least squares (OLS) regression on (1). It can be shown (see Appendix) that the same result can be obtained by a multiple regression of the factor against the sample returns to obtain the portfolio weights as done by Kandel and Stambaugh (1989).

Assume that a riskless asset exists with return r_f. Given the return on the portfolio that mimics uncertainty about state variable s, $r_s = R^Tx$, the following proposition establishes the linear pricing relation between expected returns on the N assets and the expected returns on the state-variable mimicking portfolios (see Fama, 1996; Grinblatt and Titman, 1987; and HKS, 1987).

Proposition 3. An MMV portfolio x_e that is also MV exists if and only if the following pricing relation holds:

$$E(R^e) = \beta_e E(r_e^e) + \beta_s E(r_s^e) \qquad (5)$$

where the excess returns $R^e = R - r_f$, $r_e^e = E^T x_e - r_f$ and $\begin{pmatrix}\beta_e & \beta_s^T\end{pmatrix}$ are the coefficients from regressing R^e on the returns on e and the S mimicking portfolios.

Grinblatt and Titman (1987) and HKS proved the above result. An alternate proof in the context of Fama's CAPM is in the Appendix. Note that if the ICAPM is valid (but not CAPM), e is the unique MMV and MV portfolio. It is also a tangency portfolio, i.e., a line drawn from the riskfree rate is tangent to the MV frontier at e (see Fama, 1996). Thus, a test of the ICAPM represented by problem (2) is equivalent to the test that e is the unique MMV and MV portfolio.

III. Data and Test Methodology

The data used cover the crisis and post-crisis period July 1997 to December 2001. Monthly peso-dollar rates are from the Philippine Institute of Development Studies website. Data on the stock prices and indices, and cash and stock dividends to compute simple monthly returns are from the Philippine Stock Exchange (PSE). Monthly excess returns are computed by deducting from the gross returns the monthly riskfree rate using as surrogate the treasury bill rate reported at the start of the month.

The set of risky assets include eight equally-weighted portfolios formed from 190 stocks with continuous trading during the period covered (out of 225 listed companies in mid-2001) and two market indices. The first eight portfolios are formed using the first two-digit industry classification code of the National Economic and Development Authority, the Philippine economic planning agency, as reported in Philippine Business Profiles (1998-1999). This is similar to Sweeney and Warga (1986) and Jorion (1991) except that holding companies not classified in any particular industry code are grouped together. The market indices are the PSE composite index (Phisix) and the all shares index. The more popular Phisix used in all previous asset pricing studies is a market-value-weighted index of the A and B shares of 30 representative companies from different sectors of the local bourse. Each industry is represented by a company with the highest market capitalization within the industry. The remaining 15 of the 30 companies that compose the index were also chosen based on the market capitalization. The all shares index is also a market-value-weighted index which includes all shares traded in the PSE. This index, which was only started to be reported in November 1996, is used as the market proxy in this study. The characteristics of the test assets are shown in the first three columns of Table 1.

The ICAPM test involves finding a unique benchmark portfolio e that is both MMV and MV. As Kandel and Stambaugh noted (1987), such a portfolio cannot generally be identified specifically but instead is characterized as some combination of reference portfolios. First, the portfolio that is maximally correlated with monthly movements in the peso-dollar exchange rate is obtained (the R^2 is 0.4181). Let r_M^e the excess market return using the all shares index as proxy and r_s^e the excess return on the factor mimicking portfolio, then Proposition 3 gives

$$E(r_M^e) = \beta_e E(r_e^e) + \beta_s E(r_s^e) = (1 - \beta_s)E(r_e^e) + \beta_s E(r_s^e). \tag{6}$$

Regressing excess market returns against returns on the mimicking portfolio yields an estimated beta coefficient of 0.2024. Portfolio e, with returns orthogonal to the returns on the mimicking portfolio, has mean return and variance of 0.0402 and 0.0711, respectively. To test that the MMV and MV portfolio is unique, the Wald test of equality of betas is conducted on β_s.

To test that e is MV, the procedure developed by BJS (2002) is employed. Traditional tests for MV efficiency examine the linear relation between the expected return on any asset and the beta of such asset and the expected returns on the mimicking portfolios for the state variables. For example, see Grinblatt and Titman (1987), Huberman and Kandel (1987), Kandel and Stambaugh (1987 and 1994), and Gibbons, Ross and Shanken (1989). To put this into context, the null hypothesis in these tests is that portfolio e is not MV. In the BJS test, the null is that e is MV. Thus, the two approaches complement each other. Let r_{bp} and σ_{bp}^2 be the expected return and variance of a benchmark portfolio b and σ_{ep}^2 the variance of the MV portfolio with the same expected return. Then, by Proposition 2, the benchmark portfolio is MV if and only if the BJS measure of efficiency $\lambda = \sigma_{bp}^2 - \sigma_{ep}^2 = 0$. With this as the null hypothesis, the test statistic is the BJS sample measure of efficiency $\lambda_T = \hat{\sigma}_{bp}^2 - \hat{\sigma}_{ep}^2$ where T is the number of time series observations. Applying the central limit theorem and the delta method, BJS show that $\frac{\sqrt{T}(\lambda_T - \lambda)}{\sigma_T}$ converges in distribution to the standard normal N(0, 1) as $T \to \infty$, where σ_T^2 is the asymptotic variance of $\sqrt{T}(\lambda_T - \lambda)$. BJS derive a consistent a consistent estimator of σ_T^2 that allows for heteroskedastic and autocorrelated errors in the returns data which is used in the test procedure[1].

Note that both approaches test only the coincidence of e on both the MMV and MV frontier. For the ICAPM to be valid, portfolio e must lie on the positive slope of the MMV frontier.

IV. Empirical Results

The last four columns of Table 1 show the betas (and corresponding p-values) from the regression of excess returns on the industry portfolios and market indices against excess returns on the lone state variable mimicking portfolio and on the all share index market proxy. The mimicking portfolio betas are significant at the 0.01 two-tailed significance level for three industry portfolios (manufacturing, non-bank financial intermediaries and real estate), at 0.05 for power, construction, transportation and communication and 0.10 for trade, business and other services. The market betas are significant at 0.01 significance level for all industry portfolios except for banks which is significant only at 0.10. The last two rows show the results of Wald tests of the hypotheses that the betas are equal to each other or are all zeroes. The rejection of the hypothesis that the mimicking portfolio betas are constant indicates that the corresponding state variable constraint is binding in problem (2) and there is only one point at which the MMV and MV frontier coincides. Using the celebrated GRS test due to Gibbons, Ross and Shanken (1989), the hypothesis that the intercepts are all zeroes is not rejected with p-value of 0.1690. The results provide support for the ICAPM with the foreign exchange rate as a state variable of hedging concern to investors.

The results of the BJS test are now discussed (see Table 2 and Figure 1[2]). The *ex post* MV portfolio that has the same sample mean as e has a variance of 0.0026. The reduction in variance is 0.0073. The standard error of the estimated variance reduction is 0.0092 (with 54 observations). The z-statistic (standard normal) is 0.7943. Thus, using a one-tailed test, the null hypothesis that portfolio e is MV cannot be rejected at any reasonable level of significance. However, note from Figure 1 that in mean-standard deviation space, the market proxy is located only slightly farther from the MV frontier than e. The MV portfolio that has the same sample mean as the market proxy has a variance of 0.0014. The reduction in variance is 0.0078. The standard error of the estimated variance reduction is 0.0092 and the z-statistic is 0.8511. Thus, the null hypothesis that the market proxy is MV also cannot be rejected with a very high p-value of 0.1974. However, the market proxy is not likely to be MV efficient since it is closer to the negative slope of the MV frontier (if it is MV, then the traditional CAPM holds). This is not consistent with the assumption of mean-variance optimizing behavior by investors. This result is consistent with the results of CAPM tests in the studies cited earlier.

On the other hand, the market proxy is located on the positive slope of the MMV frontier consistent with mean-variance optimizing behavior by investors. This, together with its ability to explain the observed negative market premium and the earlier results on the linear relation between expected returns, support the validity of the ICAPM as an alternative to the traditional CAPM in explaining observed cross-sectional variations in Philippine stock returns after the onset of the Asian financial crisis.

The mimicking portfolio is also tested whether it is MV. From Table 2, the variance reduction is quite large at 0.0684 (from the estimated variance of the *ex post* MV portfolio of 0.0028). The null hypothesis that the mimicking portfolio is MV is rejected (p-value of nil).

V. Conclusion

The study looks at the ability of Fama's discrete version of Merton's intertemporal CAPM to explain the cross sectional variability of Philippine stock returns after the onset of the Asian financial crisis in July 1997. Unanticipated changes in foreign exchange rate are used as an additional state variable of hedging concern to investors. The relationship between Fama's multifactor minimum-variance portfolio to the Markowitz minimum- variance portfolio is characterized in terms of the equality of the return variances for the same expected return. A test due to Basak, Jagannathan and Sun is then used to test the equality of the return variance of a derived tangency portfolio along the MMV frontier to an MV portfolio with the same sample mean return. The results do not reject the ICAPM during the period covered by the study.

Table 1 – Characteristics of Test Assets

Test Asset	No. of Stocks	Ave. Mo. Returns Mean	Variance	Intercept p-value	Exchange Rate Beta	p-value	Market Beta	p-value
Mining & Quarrying	28	0.0038	0.0266	0.8188	-0.1145	0.1182	0.9606	0.0000*
Manufacturing	32	0.0047	0.0113	0.7323	-0.1879	0.0000*	0.6405	0.0000*
Power, Const., Transp and Comm.	16	-0.0073	0.0158	0.5620	-0.0862	0.0389**	0.9580	0.0000*
Banking Institutions	14	0.0185	0.0217	0.4626	-0.1487	0.0528***	0.4185	0.0530***
Non-bank Financial Intermediaries	13	-0.0032	0.0280	0.2602	-0.2975	0.0000*	0.9585	0.0000*
Real Estate	29	-0.0086	0.0245	0.4106	-0.1692	0.0009*	1.0771	0.0000*
Trade, Business & Other Services	22	0.0367	0.0206	0.0529***	-0.0483	0.5379	0.5800	0.0088*
Holding Companies	36	0.0109	0.0271	0.8367	-0.2001	0.0783***	0.6686	0.0026*
Phisix	32	-0.0263	0.0122	0.0316**	0.0240	0.5116	1.0027	0.0000*
All Share Index	All	-0.0011	0.0092	0.1748	0.0000	1.0000	1.0000	0.0000*
Mimicking Portfolio	n.a.	0.0402	0.0711	0.4120	1.0000	0.0000*	0.0000	1.0000
Equality of coeff. test	n.a.	n.a.	n.a.	n.a.	15.7200	0.0278**	17.12	0.0166**
Zero coeff. test	n.a.	n.a.	n.a.	n.a.	127.5200	0.0000*	303.56	0.0000*

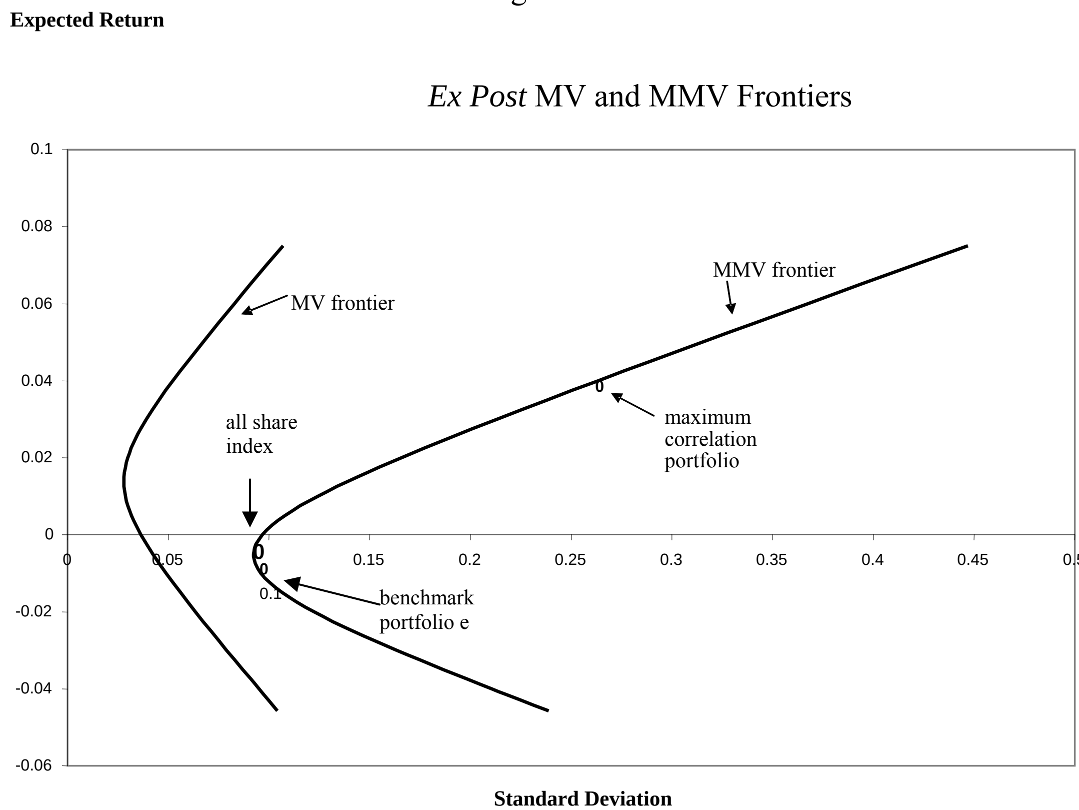
Note: Sixth and eighth columns of the last two rows are the Wald coefficient test chi-square values. The seventh and ninth are the p-values.

*   - significant at 0.01.
**   - significant at 0.05.
*** - significant at 0.10.

Table 2 – Results of BJS Test

	Derived Benchmark Portfolio	All Share Index Portfolio	Mimicking Portfolio
Mean	-0.0116	-0.0011	0.0402
Variance	0.0099	0.0092	0.0711
Variance of MV Portfolio	0.0026	0.0014	0.0028
Variance Reduction	0.0073	0.0078	0.0684
Standard Error	0.0092	0.0092	0.0045
z-statistic	0.7943	0.8511	15.0862
p-value (one-tail)	0.2152	0.1974	0.0000

Figure 1


Expected Return
Ex Post MV and MMV Frontiers
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
MV frontier
MMV frontier
all share index
maximum correlation portfolio
benchmark portfolio e
Standard Deviation

Appendix

Proof of Proposition 1

The Lagrangean of optimization problem (2) is

$$L = \tfrac{1}{2}\, x^T V x - \gamma_1(B^T x - b_e^T) - \gamma_2(E^T x - r_p) - \gamma_3(x^T i - 1)$$

where γ_1 is a S x 1 vector of Lagrange multipliers and γ_2 and γ_3 are scalar Lagrange multipliers. The first order condition is[3]

(A.1) $Vx = (B\gamma_1 + \gamma_2 E + \gamma_3 i)$

This with the constraints (2.a) to (2.c) yields the MMV portfolio

(A.2) $x_e = V^{-1}(B\ E\ i)\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}$

Premultiplying by $(B\ E\ i)^T$ yields

$$(B\ E\ i)^T x_e = (B\ E\ i)^T V^{-1}(B\ E\ i)\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}$$

Let the (S+2) x (S+2) matrix

$$A = (B\ E\ i)^T V^{-1}(B\ E\ i)$$

This gives

$$A^{-1}(B\ E\ i)^T x_e = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}$$

$$A^{-1}\begin{pmatrix}B^T x_e\\ E^T x_e\\ i^T x_e\end{pmatrix} = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} \qquad \text{or}$$

(A.3) $$\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = A^{-1}\begin{pmatrix}b_e\\ r_p\\ 1\end{pmatrix}$$

Substituting this into (A.2)

$$x_e = V^{-1}\ (B\ E\ i)\ A^{-1}\begin{pmatrix}b_e\\ r_p\\ 1\end{pmatrix}$$

Note that V^{-1} is positive definite and

$$A = (B\ E\ i)^T V^{-1}(B\ E\ i) = \begin{pmatrix}D & e & f\\ e^T & a & b\\ f^T & b & c\end{pmatrix} = \begin{pmatrix}D & G^T\\ G & \widetilde{A}\end{pmatrix}$$

where $a = E^TV^{-1}E$, $b = E^TV^{-1}i$, $c = i^TV^{-1}i$, $D = B^TV^{-1}B$, $e = B^TV^{-1}E$, and $f = B^TV^{-1}i$ and $G^T = (e\ f)$ and $\widetilde{A} = \begin{pmatrix}a & b\\ b & c\end{pmatrix}$. Note that A, $\widetilde{A}$ and D are positive definite matrices.

The variance of portfolio return is

$$\sigma_{ip}^2 = x_e^T V x_e = \begin{pmatrix}b_e^T & r_p & 1\end{pmatrix} A^{-1}\begin{pmatrix}B^T\\ E\\ i\end{pmatrix} V^{-1}\begin{pmatrix}B & E & i\end{pmatrix} A^{-1}\begin{pmatrix}b_e\\ r_p\\ 1\end{pmatrix}$$

$$= \begin{pmatrix}b_e^T & r_p & 1\end{pmatrix} A^{-1}\begin{pmatrix}b_e\\ r_p\\ 1\end{pmatrix} = \begin{pmatrix}b_e^T & \widetilde{R}^T\end{pmatrix} A^{-1}\begin{pmatrix}b_e\\ \widetilde{R}\end{pmatrix}.$$

where $\widetilde{R}^T = \begin{pmatrix}r_p & 1\end{pmatrix}$. Note that in the Markowitz efficient frontier (see Merton, 1972 or Roll, 1977), the portfolio return variance is

$$\sigma_{mp}^2 = \begin{pmatrix}r_p & 1\end{pmatrix}\widetilde{A}^{-1}\begin{pmatrix}r_p\\ 1\end{pmatrix} = \widetilde{R}^T\widetilde{A}^{-1}\widetilde{R}$$

where $\widetilde{A}$ is defined as above.

Let the inverse of A be partitioned as

$$A^{-1} = \begin{pmatrix} M & N^T \\ N & O \end{pmatrix}$$

where the partitions M (S x S), N (2 X S) vector and O (2 x 2) will be defined later. Then the portfolio return variance is

$$\sigma_{ip}^2 = \begin{pmatrix} b_e^T & r_p & 1 \end{pmatrix} A^{-1} \begin{pmatrix} b_e \\ r_p \\ 1 \end{pmatrix} = \begin{pmatrix} b_e^T & \tilde{R}^T \end{pmatrix} \begin{pmatrix} M & N^T \\ N & O \end{pmatrix} \begin{pmatrix} b_e \\ \tilde{R} \end{pmatrix}.$$

Then,

(A.4) $\sigma_p^2 = b_e^T M b_e + 2b_e^T N^T \tilde{R} + \tilde{R}^T O \tilde{R}$.

The elements of A^{-1} are (see, for example, Greene, 2000, p. 34)

(A.5) $M = D^{-1}(I + G^T OGD^{-1}) = (D - G^T \tilde{A}^{-1} G)^{-1}$

(A.6) $N^T = - D^{-1} G^T O$

(A.7) $O = (\tilde{A} - GD^{-1}G^T)^{-1}$

I in (A.5) is the identity matrix. Note that M is positive definite. Now, using the results on the updating of matrix inverses from Greene, p. 32

(A.8) $O = \tilde{A}^{-1} + \tilde{A}^{-1} G \left(D - G^T \tilde{A}^{-1} G\right)^{-1} G^T \tilde{A}^{-1}$.

Thus,

$$\tilde{R}^T O \tilde{R} = \tilde{R}^T \tilde{A}^{-1} \tilde{R} + (\tilde{R}^T \tilde{A}^{-1} G)(D - \tilde{G}^T \tilde{A}^{-1} G)^{-1} (G^T \tilde{A}^{-1} \tilde{R}).$$

From (A.5) to (A.7)

$$N^T = -D^{-1} G^T (\tilde{A}^{-1} + \tilde{A}^{-1} G(D - G^T \tilde{A}^{-1} G)^{-1} G^T \tilde{A}^{-1})$$

Thus,

$$2b_e \tilde{R}^T N = -2b_e^T D^{-1} (G^T \tilde{A}^{-1} \tilde{R} + G^T \tilde{A}^{-1} G(D - G^T \tilde{A}^{-1} G)^{-1} (G^T \tilde{A}^{-1} \tilde{R}))$$

$$= -2b_e^T D^{-1} [I + G^T \tilde{A}^{-1} G(D - G^T \tilde{A}^{-1} G)^{-1}](G^T \tilde{A}^{-1} \tilde{R}))$$

$$= -2b_e^T(D - G^T\widetilde{A}^{-1}G)^{-1}(G^T\widetilde{A}^{-1}\widetilde{R})$$

Finally, using (A.5) and plugging the above result into (A.4)

$$\sigma_{ip}^2 = \widetilde{R}^T\widetilde{A}^{-1}\widetilde{R} + b_e^T M b_e - 2b_e^T M(G^T\widetilde{A}^{-1}\widetilde{R}) + (\widetilde{R}^T\widetilde{A}^{-1}G)M(G^T\widetilde{A}^{-1}\widetilde{R})$$

$$= \sigma_{mp}^2 + (b_e - G^T\widetilde{A}^{-1}\widetilde{R})^T M(b_e - G^T\widetilde{A}^{-1}\widetilde{R})$$

$$\geq \sigma_{mp}^2 .$$

The last inequality follows since M is positive definite and thus, the last term in the second line above is non-negative.

Proof of Proposition 2

If M is positive definite and x is on the MMV frontier, then $\sigma_{mp}^2 = \sigma_{ip}^2$ implies that $b_e = G^T\widetilde{A}^{-1}\widetilde{R}$. This also follows from setting $\gamma_1 = 0$ in (A.3) making constraint (2.a) in problem (2) redundant. This in turn means that

$$B^T x = \left(B^T V^{-1}E \quad \widetilde{B}^T V^{-1} i\right)\widetilde{A}^{-1}\widetilde{R} = B^T V^{-1}(E \quad i)\widetilde{A}^{-1}\left(r_p \quad 1\right)^T .$$

Since B is of full rank, this means that

(A.9) $\quad x_0 = V^{-1}(E \quad i)\widetilde{A}^{-1}\left(r_p \quad 1\right)^T$

is a solution to $B^T x = b_e$ and on the MV efficient frontier (see Merton, 1972 or Roll, 1977). Existence is shown by noting that $E^T x_0 = r_p$.

To prove the corollary, it suffices to show that if i is in the column space of B, any portfolio such that $E^T x = r_p$ for arbitrary r_p also implies that $b_e = G^T\widetilde{A}^{-1}\widetilde{R}$. That is,

$$b_e = B^T V^{-1}(E \quad i)\widetilde{A}^{-1}\left(r_p \quad 1\right)^T = B^T V^{-1}(E \quad i)\widetilde{A}^{-1}(E \quad i)^T x .$$

If i is in the column space of B, then there exists a unique vector δ such than $\delta^T B^T = i^T$. Thus,

$$\delta^T b_e = i^T V^{-1}(E \quad i)\widetilde{A}^{-1}(E \quad i)^T x = i^T x = 1$$

and

$$\delta^T b_e = \delta^T B^T x = i^T x = 1$$

Proof of Proposition 3

The necessity proof is from Fama (1996). Existence of a portfolio that is both MMV and MV is assured by Proposition 2. With a riskless asset with return r_p, constraint (2.c) is dropped and (2.b) is replaced by

(2.b') $$(E(R^e) - ir_f)^T x = r_p^e$$

where $R^e = R - r_f$ and $r_p^e = r_p - r_f$. The Lagrangean becomes

$$L = ½\, x^T V x - \gamma_1 (B^T x - b_e^T) - \gamma_2 (E(R^e)^T x - r_p^e)$$

with first order condition

$$Vx = (B\gamma_1 + \gamma_2 E(R^e))$$

This with the constraints (2.a) and (2.b') yields the following

$$x_e = V^{-1} B \lambda_1 + V^{-1} E(R^e) \lambda_2$$

$$E(R^e) = \tfrac{1}{\lambda_2} V x_e - VV^{-1} B \tfrac{\lambda_1}{\lambda_2}$$

Replacing $V^{-1}B$ with $X_s = [x_1| x_2|\ldots|x_S]$, the N x S matrix of weights of the S mimicking portfolios (appropriately rescaling λ_1), yields

(A.10) $$E(R^e) = \tfrac{1}{\lambda_2} V x_e - V X_s \tfrac{\lambda_1}{\lambda_2} = \begin{pmatrix} Vx_e & VX_s \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_2} \\ \frac{\lambda_1}{\lambda_2} \end{pmatrix}.$$

The excess returns on e and the S mimicking portfolios can now be expressed as

(A.11) $$\begin{pmatrix} E(r_e^e) \\ E(R_s^e) \end{pmatrix} = \begin{pmatrix} x_e & X_s \end{pmatrix}^T \begin{pmatrix} Vx_e & VX_s \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma_2} \\ \frac{\gamma_1}{\gamma_2} \end{pmatrix} = V_{es} \begin{pmatrix} \frac{1}{\gamma_2} \\ \frac{\gamma_1}{\gamma_2} \end{pmatrix}$$

where V_{es} is the (S+1) x (S+1) covariance matrix of excess returns on e and the S mimicking portfolios. From (A.11), the vector of Lagrange multipliers is now derived as

$$\begin{pmatrix} \frac{1}{\gamma 2} \\ \frac{\gamma 1}{\gamma 2} \end{pmatrix} = V_{es}^{-1} \begin{pmatrix} E(r_e^e) \\ E(R_s^e) \end{pmatrix}.$$

Substituting this into (A.10) yields

$$E(R^e) = \begin{pmatrix} Vx_e & VX_s \end{pmatrix} V_{es}^{-1} \begin{pmatrix} E(r_e^e) \\ E(R_s^e) \end{pmatrix}$$

leading to the desired result

$$\text{(A.12)} \quad E(R^e) = \begin{pmatrix} \beta_e & \beta_s^T \end{pmatrix} \begin{pmatrix} E(r_e^e) \\ E(R_s^e) \end{pmatrix}$$

where $\begin{pmatrix} \beta_e & \beta_s^T \end{pmatrix}$ is the N x (S+1) of coefficients from regressing R on the returns on e and the S mimicking portfolios.

To show sufficiency, assume that (A.12) holds. Let $\beta_s^T = VX_s V_{es}^{-1} = 0$. This implies that $VX_s = 0$ and from (A.10)

$$Vx_e = \gamma_2 E(R^e)$$

is the first order condition for optimization problem (2) without constraint (2.b'). Thus, e is an MV portfolio.

OLS Estimate of the Mimicking Portfolio

The tangency point of a line drawn from the origin to the feasible set of optimization problem (4) in (b_{es}, σ_p) space is $x_s = V^{-1}b_s/i^TV^{-1}b_s$. Let the sample returns be expressed in deviations from sample means, $\tilde{r} = \tilde{R} - \overline{R}$ (where '~' denotes the sample realizations of the random variables), and $\hat{V}^{-1} = \tilde{r}^T\tilde{r}$ is the sample covariance matrix of returns. An estimate of the tangency point is $\hat{x}_s = h\hat{V}^{-1}\hat{b}_s$ where the vector $\hat{b}_s$ consists of the slope coefficients of an OLS regression of the columns of $\tilde{r}$ against $\tilde{k}_s$ and h is a constant of proportionality to ensure that $i^T\hat{x}_s = 1$. Then,

$$\hat{x}_s = h\hat{V}^{-1}\hat{b}_s = h(\tilde{r}^T\tilde{r})^{-1}(\tilde{k}_s^T\tilde{k}_s)^{-1}\tilde{r}^T\tilde{k}_s = h'\hat{\Lambda}$$

where now $\hat{\Lambda}$ is the coefficient matrix of the multivariate regression of $\tilde{k}_s$ against $\tilde{r}$ and $h' = h(\tilde{k}_s^T \tilde{k}_s)^{-1}$ is the new proportionality constant.

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ENDNOTES

[1] The covariance matrix under the delta method is composed of covariances between returns and transformations of returns data. The covariance matrix is estimated using the Newey-West (1987) method. Since monthly returns are generally AR(1), a maximum lag of 3 is used.
[2] Figure 1 has standard deviation as the horizontal axis instead of the variance, following traditional practice.
[3] Since V is positive definite and the constraints are linear, the second order condition for minimization is met.